Stability of Naturally Bounded Nonlinear Systems

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Liapunov's direct method is used to obtain regions of asymptotic stability for nonlinear systems with natural boundaries. In some cases this leads to larger stable regions of interest than if the bounds had been neglected. With account taken of the natural boundaries, a design method is formulated to find controller gains that give stable responses for anticipated disturbances. To compare the stability effects of different controllers and their gains, the change in the shape of the Liapunov function must be considered as well as the change in size of the region in which its time derivative is negative definite. Neglect of changes in the shape of the Liapunov function has led to misleading conclusions in previous work.

Liapunov's direct (or second) method can be used to predict the stability behavior of nonlinear systems without solving actually the differential equations that describe the systems. For a general discussion of Liapunov functions (LF) and the validity of the direct method, several references are available (*4, 5, 7, 9*).

Until recently, work published in this area in the chemical engineering literature had been sparse. In a series of articles Berger and Perlmutter (*1* to *3*) discussed the application of Krasovskii's theorem to the determination of Liapunov functions for perfectly mixed, continuous flow, stirred-tank reactors (CSTAR). A note to their work was added by Luecke and McGuire (*10*). Lapidus and co-workers adapted the second method to an extraction system (*6*) and a procedure for determining limit cycles (*8*). Amundson and co-workers (*12*) formulated a technique for deriving Liapunov functions for reaction systems. Wei (*13*) used the direct method to determine stability in a catalyst particle.

An adaptation of Liapunov's theorem given by LaSalle and Lefschetz (*7*) is especially useful in the design of controllers for nonlinear autonomous systems. An autonomous system may be defined by the matrix equations

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$$
$$\mathbf{f}(\mathbf{0}) = \mathbf{0} \qquad (1)$$

where **X** is a generalized state column vector that is identically zero when the system is at steady state. If there exists a positive definite scalar function $V(\mathbf{X})$ with continuous first partial derivatives and negative definite time derivative within the bounded region defined by $V(\mathbf{X}) < K$, then the system is asymptotically stable to disturbances within that region, and $V(\mathbf{X})$ is called a Liapunov function for the dynamic system given by Equation (1) in the bounded region. Applied to control system design, this method gives positive assurance of stability as long as disturbances are within the well-defined region about the steady state. Linear approximations of nonlinear systems, however, only guarantee stability in a small, undefined region about the steady state. A disadvantage of the direct method is that it gives only sufficient conditions for stability. The failure of a particular Liapunov function to define a region as stable does not necessarily imply instability for that region.

In the applications of Liapunov's direct method to control system design, most emphasis has been placed on extending the regions of asymptotic stability (RAS) to make them as large as possible. In reference *1* this was done by investigating the effect of the controller gain on the size of the region where $\dot{V}(\mathbf{X}) < 0$. There are practical cases, however, where increasing the size of the RAS does not correspond to improved stability limits of the system but actually leads to poorer limits. This may occur in systems where natural boundaries are present and the RAS overlaps the boundary, thus including a portion of the phase space which is physically unattainable. In this paper natural boundaries are considered and their effects on Liapunov's direct method are examined.

EFFECTS OF FEEDBACK CONTROL

Liapunov's direct method can be used to examine the effect of different feedback controller modes and gains on the stability of the nonlinear system described by Equation (1), but in comparing different systems, one must exercise some caution. To demonstrate this it is useful to consider a particular method of obtaining Liapunov functions, the application of Krasovskii's theorem (*1* to *3*). The theorem and applications have been described in the literature (*1, 3* to *5, 11*) and no exposition will be given here.

If we apply Krasovskii's theorem to the system defined by Equation (1), a Liapunov function is given by

$$V(\mathbf{X}) = \mathbf{f}^T(\mathbf{X})\mathbf{B}\mathbf{f}(\mathbf{X}) \qquad (2)$$

where **B** is any positive definite, symmetric matrix. A convenient choice of **B** is the unit matrix $\mathbf{B} = \mathbf{I}$, and

$$V(\mathbf{X}) = \mathbf{f}^T(\mathbf{X})\mathbf{f}(\mathbf{X}) \qquad (3)$$

In this case it can be shown (*1, 5*) that

$$\dot{V}(\mathbf{X}) = \mathbf{f}^T(\mathbf{X})[\mathbf{F}^T(\mathbf{X}) + \mathbf{F}(\mathbf{X})]\mathbf{f}(\mathbf{X}) \qquad (4)$$

where **F** is the Jacobian matrix of **f**(**X**). In order for $V(\mathbf{X})$ to be negative definite, it is sufficient that $(\mathbf{F}^T + \mathbf{F})$ be negative definite, or that $-(\mathbf{F}^T + \mathbf{F})$ be positive definite.

Therefore a region in which $V(\mathbf{X})$ is negative definite and the system is asymptotically stable to disturbances can be found by determining the Sylvester inequalities of

$-(\mathbf{F}^T + \mathbf{F})$. The stable region can be defined in general by the boundary $\phi(\mathbf{X}) = 0$, which is obtained from the Sylvester inequalities. This boundary is termed a *separatrix* because it divides the phase plane into the regions where $(\mathbf{F}^T + \mathbf{F})$ is negative definite and where this condition does not hold. However, as was pointed out by Luecke and McGuire (10), $\phi(\mathbf{X}) = 0$ is not necessarily the same curve as $\dot{V}(\mathbf{X}) = 0$. Then the largest possible region of asymptotic stability (RAS) for a particular LF derived by this method will be defined by a $V(\mathbf{X}) = K$ contour which just intercepts the boundary $\phi(\mathbf{X}) = 0$, but does not extend beyond it into the region of indeterminate behavior.

In choosing control systems, an obvious criterion appears to be one where the boundary $\phi(\mathbf{X}) = 0$ is as far from the steady state as possible. But this is not the only criterion that should be applied, for in changing control parameters, the Liapunov function also is changed, and the change in its shape may be significant. As an example the familiar continuous, stirred-tank reactor will be considered.

The energy and mass balances give

$$v\dot{C} = q(C_o - C) - vr \tag{5}$$

$$\rho v C_p \dot{T} = \Delta H v r - Ua(T - T_j) + \rho q C_p (T_o - T) \tag{6}$$

The concentration C refers to the species reacting and r is the rate at which it is removed by chemical reaction. By using transformations to first normalize the equations and then to translate them to a form where the variables represent departures from a steady state (*1*), one obtains

$$\dot{X}_1 = -\alpha X_1 - \frac{R}{C_o} \tag{7}$$

$$\dot{X}_2 = -(\alpha + \beta) X_2 + \frac{R}{C_o} \tag{8}$$

where

$$\alpha = \frac{q}{v} \quad \beta = \frac{Ua}{\rho C_p v}$$

$$x_1 = \frac{C}{C_o} \quad x_2 = \frac{\rho C_p T}{\Delta H C_o}$$

$$\mathbf{X} = \mathbf{x} - \mathbf{x}_s$$

$$R = r - r_s$$

The subscript s denotes steady state quantity.

Addition of feedback control will introduce new terms to Equations (7) and (8). For example, T could be the controlled variable and T_j, the temperature in a heat exchanger, the manipulated variable. In this case Equation (8) would become

$$\dot{X}_2 = -(\alpha + \beta) X_2 + \frac{R}{C_o} + \beta X_j \tag{9}$$

where $X_j = x_j - x_{js}$ = some function of X_2 depending on the mode of control.

If T_o instead of T_j is the manipulated variable, Equation (8) becomes

$$\dot{X}_2 = -(\alpha + \beta) X_2 + \frac{R}{C_o} + \alpha X_o \tag{10}$$

and in this case X_o is a function of X_2. For proportional control

$$X_o = -K_{po} X_2 \tag{11}$$

or

$$X_j = -K_{pj} X_2 \tag{12}$$

and the system described by Equations (7) and (9), that is, manipulating T_j, will be analogous to the one described by (7) and (10). They will be identical in the special case where $K_{pj} = \frac{\alpha}{\beta} K_{po}$. Thus manipulating T_j, heat exchanger temperature, is analogous to manipulating T_o, inlet temperature, and we will arbitrarily choose to manipulate T_j.

One can now apply Sylvester's theorem to obtain a Liapunov function, $V(\mathbf{X})$, and the region in which $\dot{V}(\mathbf{X})$ is negative definite. From Equation (3) it follows that $V(\mathbf{X})$ is given by

$$V(\mathbf{X}) = \left(-\alpha X_1 - \frac{R}{C_o} \right)^2 + \left(-(\alpha + \beta) X_2 + \frac{R}{C_o} - \beta K_{pj} X_2 \right)^2 \tag{13}$$

The region in which $\dot{V}(\mathbf{X})$ is negative definite may be obtained by examining the Sylvester inequalities of $-(\mathbf{F}^T + \mathbf{F})$. In this example the matrix $-(\mathbf{F}^T + \mathbf{F})$ is given by

$$-(\mathbf{F}^T + \mathbf{F}) = \begin{bmatrix} 2\left(\alpha + \frac{\partial R}{C_o \partial X_1}\right) & \frac{\partial R}{C_o \partial X_2} - \frac{\partial R}{C_o \partial X_1} \\ \frac{\partial R}{C_o \partial X_2} - \frac{\partial R}{C_o \partial X_1} & 2\left(\alpha + \beta - \frac{\partial R}{C_o \partial X_2} + \beta K_{pj}\right) \end{bmatrix} \tag{14}$$

and its Sylvester inequalities are

$$2\left(\alpha + \frac{\partial R}{C_o \partial X_1}\right) > 0 \tag{15}$$

$$4\left(\alpha + \frac{\partial R}{C_o \partial X_1}\right)\left(\alpha + \beta - \frac{\partial R}{C_o \partial X_2} + \beta K_{pj}\right) - \left(\frac{\partial R}{C_o \partial X_2} - \frac{\partial R}{C_o \partial X_1}\right)^2 > 0 \tag{16}$$

If a typical reaction rate such as

$$r = A \exp(-E/R_g T) C^n = k C^n \tag{17}$$

is assumed, and appropriate transformation and partial differentiation are performed, inequality (15) becomes

$$\left(\alpha + \frac{nr}{C_o x_1}\right) > 0 \tag{18}$$

None of the terms on the left-hand side of (18) is negative, so this inequality is always satisfied for $n \geqq 0$. After introduction of the reaction rate of (17), inequality (16) becomes

$$4\left(\alpha + \frac{nr}{C_o x_1}\right)\left(\alpha + \beta - \frac{Qr}{C_o x_2^2} + \beta K_{pj}\right) - \left(\frac{Qr}{C_o x_2^2} - \frac{nr}{C_o x_1}\right)^2 > 0 \tag{19}$$

where $Q = E\rho C_p / R_g \Delta H C_o$.

The boundary of the region where $V(\mathbf{X})$ is negative definite is determined by setting the left-hand side of (19), which may be denoted $\phi(\mathbf{x})$, equal to zero. Any

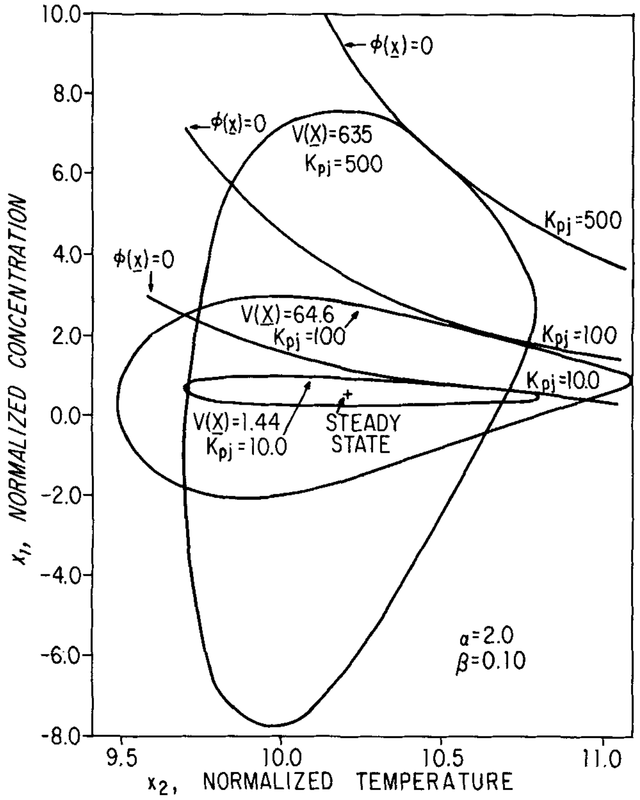

Fig. 1. Regions of asymptotic stability for proportional control of a stirred-tank reactor without consideration of the natural boundary.

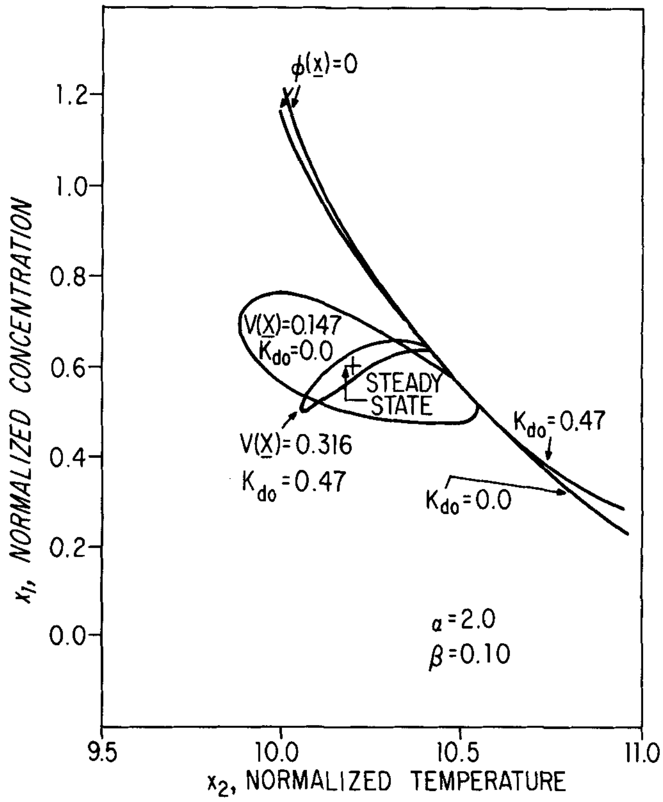

Fig. 2. Regions of asymptotic stability for derivative control, as given in reference (1), of a stirred-tank reactor.

region that is both within a $V(\mathbf{X}) = K$ contour and on the stable side of the separatrix $\phi(\mathbf{x}) = 0$ is a region of asymptotic stability.

Using the data obtained from (*1*) for a first-order ($n = 1$) exothermic reaction, we have plotted boundaries, $\phi(\mathbf{x}) = 0$, and maximum regions of asymptotic stability for this LF in Figure 1 with the proportional control constant K_{pj} as a parameter. As noted by Berger and Perlmutter (*1*), an increase in K_{pj} strengthens inequality (19) and expands the RAS, moving it further away from the steady state point. However, this interpretation is incomplete and may be misleading since the shape of the contour $V(\mathbf{X}) = K$ also changes with K_{pj}. For the example used, the range of temperature in which the system is asymptotically stable first increases but then goes through a maximum and decreases as K_{pj} is increased further. Therefore, the position of the boundary $\phi(\mathbf{x}) = 0$ is not the only consideration in deciding on a controller gain, for in choosing an optimum setting, the stable limits on a particular variable may be far more important than the absolute size of the RAS.

Derivative control as given in (*1*) also appears to increase stability if one is guided only by its effect on the constraint, $\phi(\mathbf{x}) = 0$, but in this case the RAS predicted actually shrinks because of the change brought about in the Liapunov function. This is shown in Figure 2 where two different values of the derivative control constant K_{do} are used as parameters. For this case, identical with one in the literature (*1*), the inlet temperature was the manipulated variable.

The curves shown in Figures 1 and 2 were generated with a short digital program solved on the IBM 7040-7094 system at the Computer Center of the University of Washington.

Again it should be noted that the analysis using Liapunov's direct method provides only sufficient conditions for stability; the system may in fact be globally stable even when a bounded RAS is predicted. Complete knowledge of the actual transient behavior of the reactor would be valuable for comparing the approximate RAS with the true RAS, but to establish this knowledge, even in the case of global stability, requires extensive computation since both a unique steady state and the absence of limit cycles must be proven. Bendixon's theorem (*4*) was applied in testing for limit cycles in the example above, but results were inconclusive. The actual transient behavior is not necessary, however, for extending and improving applications of Liapunov's direct method.

LIAPUNOV FUNCTIONS FOR NATURALLY BOUNDED SYSTEMS

Some systems are naturally constrained by boundaries due to their physical nature. For example, if the reactor described in the previous section suffered only temperature disturbances, the concentration of reactant in the vessel could never exceed the concentration in the feed, nor could it be negative. Thus the range of the variable x_1 is

$$0 \leqq x_1 \leqq 1.0$$

If large disturbances in the feed concentration were anticipated, some estimate of the upper bound on x_1 would probably be available. In such cases a change in the Liapunov function and the separatrix $\phi(\mathbf{X}) = 0$, could lead to an overall increase in the RAS, but an actual decrease in the RAS within the true region of interest, that is, the region where the variables are within their natural bounds. Conversely, an overall decrease in the RAS could be accompanied by an actual improvement of the stability limits for the real system. In these cases, ex-

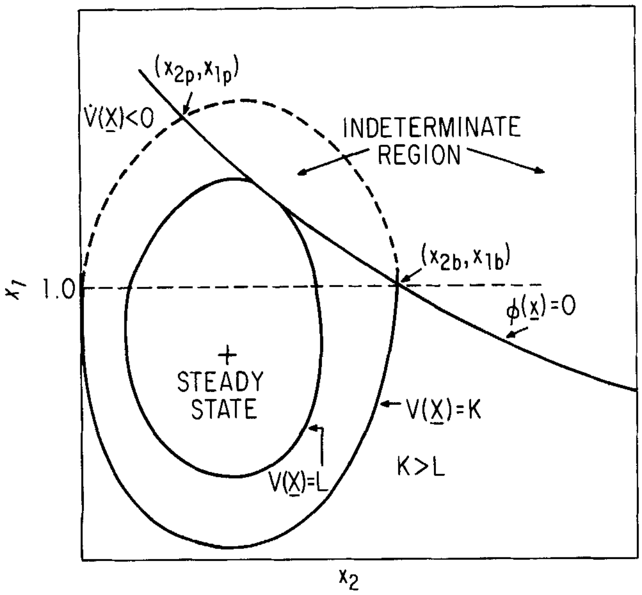

Fig. 3. Natural boundary improves the region of asymptotic stability.

amining only the RAS while ignoring the natural boundaries is superficial.

On the other hand in systems with natural bounds, the largest contour, $V(\mathbf{X}) = K$, can cross the separatrix into the region of indeterminate stability, if the indeterminate region so enclosed is outside the natural boundary of the system. This follows since no trajectory of a real system can cross the natural boundary. This can be seen by examining the differential equations which describe the system. In Equation (5), for example, the derivative of concentration with time must be zero or negative at $C = C_o$, and the concentration must either decrease or remain constant.

Two hypothetical cases are illustrated in Figures 3 and 4. In Figure 3 the contour $V(\mathbf{X}) = L$ encloses the RAS that would be found by ignoring any natural boundary. The larger contour $V(\mathbf{X}) = K$ would be rejected if only the criterion that $\dot{V}(\mathbf{X}) < 0$ was applied. If, however, the system is subject to the natural bound that $0 \leqq x_1 \leqq 1.0$, the RAS is bounded by $x_1 = 1.0$, $x_1 = 0.0$, and $V(\mathbf{X}) = K$. In this case consideration of the natural bound extends the RAS in the region of interest.

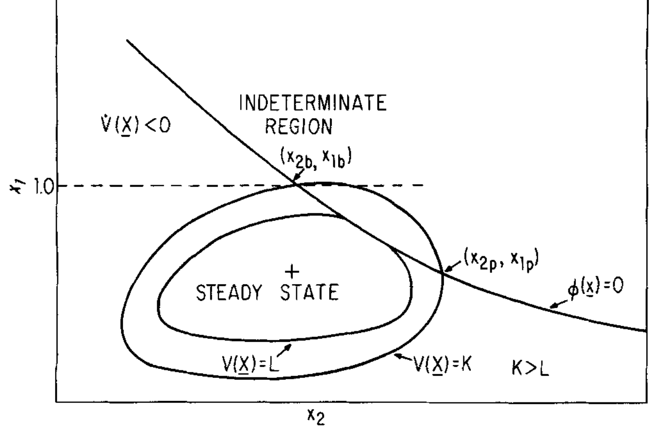

Fig. 4. Natural boundary does *not* improve the region of asymptotic stability.

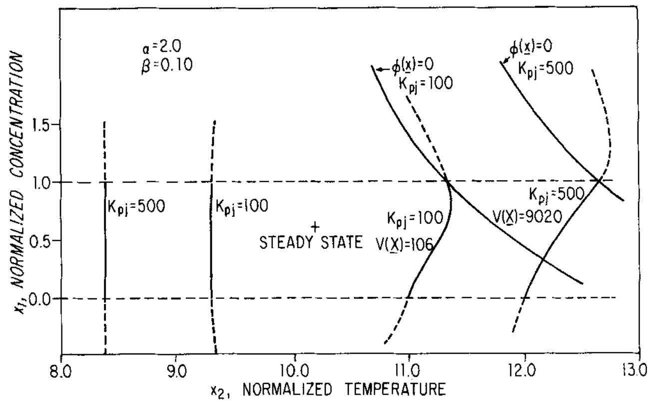

Fig. 5. Regions of asymptotic stability for proportional control of a stirred-tank reactor utilizing the natural boundary.

Figure 4 depicts a case where the natural bound will not ameliorate the RAS. Although the figures have been plotted in terms of reduced variables to emphasize the natural boundary, plots with disturbance variables would give equivalent results.

Consideration of natural boundaries may modify the interpretation of what happens when feedback control is added to a process. Figure 1 showed that increasing the gain of a proportional controller on a stirred-tank reactor led first to an increase followed by a decrease in stable range of temperature disturbances. But no consideration for natural bounds was made. If it is assumed that the concentration variable x_1 is constrained to values such that $0 \leqq x_1 \leqq 1.0$, one can analyze the system differently. Figure 5 shows a segment of the entire phase plane where attention has been confined to the region of interest (data the same as for Figure 1). The separatrices $\phi(\mathbf{x}) = 0$ are plotted for two values of controller gain, $K_{pj} = 100$ and $K_{pj} = 500$, and are the same as the separatrices of Figure 1. The contour $V(\mathbf{X}) = K$ may be chosen differently than before, however, since the only concern is that it does not cross the separatrix in the region $0 \leqq x_1 \leqq 1.0$ but may cross it outside this region. The largest RAS that lies within the natural boundary for each controller setting is defined by the contours shown in Figure 5, and for this case it is apparent that the RAS in the region of interest increases with controller gain, and the range of stable temperature disturbances increases significantly with a change in K_{pj}. In the case considered, these large disturbances might well cause a proportional controller to become saturated. Nevertheless, this example illustrates the advantage of using natural boundaries whenever possible.

Care should be taken to ensure that no region is entered where any of the first partial derivatives of $V(\mathbf{X})$ becomes discontinuous. This would invalidate the theorems upon which the analysis is based. [For reaction rates of order n, where $0 < n < 1.0$ the partial derivative of $V(\mathbf{X})$ with respect to X_1 is discontinuous at $x_1 = 0.0$, but as noted by Berger and Perlmutter (*3*), no trajectory starting above zero concentration can ever reach zero.] The possibility of a disturbance starting at zero concentration seems remote. Therefore, stability information for these reaction orders ($0 < n < 1.0$) can be obtained in the same manner as when $n \geqq 1.0$.

Design for Stability

From the foregoing properties of systems having natural bounds, a systematic procedure for designing control systems or checking asymptotic stability limits of existing systems can evolve because $V(\mathbf{X})$ and $\phi(\mathbf{X})$ contain control parameters. To this end, a few pertinent observations will be made.

By comparison of Figures 3 and 4, which represent second-order nonlinear systems, it becomes evident when an upper natural boundary can be utilized as an aid in enlarging the RAS of interest. In both cases $V(\mathbf{X}) = K$ has been constructed to intersect $\phi(\mathbf{x}) = 0$ where it crosses the boundary $x_1 = 1.0$. In Figure 3 the enlarged region does not allow a trajectory to cross $\phi(\mathbf{x}) = 0$, while in Figure 4 it does, and no stability information can be obtained from the enlarged area. For convenience, the two curves $\phi(\mathbf{x}) = 0$ and $V(\mathbf{X}) = K$ will be referred to as ϕ and V, respectively. The point of intersection of ϕ with the natural bound is b. The other point of intersection between the constraint ϕ and the LF V is p.

If there are no more than two intersections between ϕ and V and ϕ intersects the natural bound at no more than one point, a design procedure can be formulated. If ϕ is monotonic and V is convex, these criteria are assured. For an upper natural bound, for example, $x_1 = 1.0$, the two intersection points are compared. If $x_{1b} < x_{1p}$, the method will give increased stability information because the LF encircles part of ϕ inside the region of interest. If $x_{1b} > x_{1p}$, the method will not give increased stability information because the LF will encircle part of ϕ inside the region of interest. For a lower bound the reverse is true. If the steady state is stable, no intersections between V and ϕ corresponds to a RAS within all of the Liapunov function, and one intersection, a point of tangency, is equivalent to no intersections for that LF.

It is first necessary to determine that the steady state is indeed stable for the LF obtained from Krasovskii's theorem. This is accomplished by substituting the coordinates of the steady state into the constraint. If $\phi(x_{1s}, x_{2s}) > 0$, the steady state is stable. If $\phi(x_{1s}, x_{2s}) \leqq 0$, search for a better LF.

If it is desired to estimate the size of the largest stable disturbance for a given controller gain, solve the equations, $V(\mathbf{X}) = K$ and $\phi(\mathbf{x}) = 0$, simultaneously at the boundary. This point can be used to obtain K, and then check for the other point of intersection. With $x_{1b} < x_{1p}$ assumed, the largest LF (by this method) has been found.

If the reverse is desired, that is, given some maximum disturbance at a boundary, find the smallest controller gain which will cause it to return to the steady state, substitute this point directly into $\phi(\mathbf{x}) = 0$, and obtain K_p. More generally other disturbances besides those at a bound could be used, but the algebra becomes more complicated. Then find the two intersections of V and ϕ to verify the proper relationship between x_{1b} and x_{1p}. Plot a few points of $V(\mathbf{X})$ to show whether the stable disturbances at other x_1 are adequate.

To illustrate this procedure, assume that one wants to find the smallest gain of a proportional feedback controller manipulating reactor coolant temperature that will ensure stability for a 54°F. disturbance at the bound, $x_1 = 1.0$.

$$x_{2b} = x_{2s} + \frac{(54)(\rho)(C_p)}{(\Delta H)(C_o)} = 11.2$$

The constraint formed from the Sylvester inequality of the Jacobian matrix for this system is given by setting inequality (19) equal to zero. The Liapunov function is obtained from Equation (13), and the values of the CSTAR parameters are obtained from reference *1*.

From $\phi(x_{1b}, x_{2b}) = 0$, $K_{pj} = 77.4$ This is equivalent to $K_{po} = 3.87$ for proportional control manipulating the feed temperature.

Substitution of x_{1b}, x_{2b}, and K_{pj} into Equation (13) gives $V(\mathbf{X}) = 59.1$ To check the intersections of ϕ and V, solve $\phi(\mathbf{x}) = 0$ for x_1.

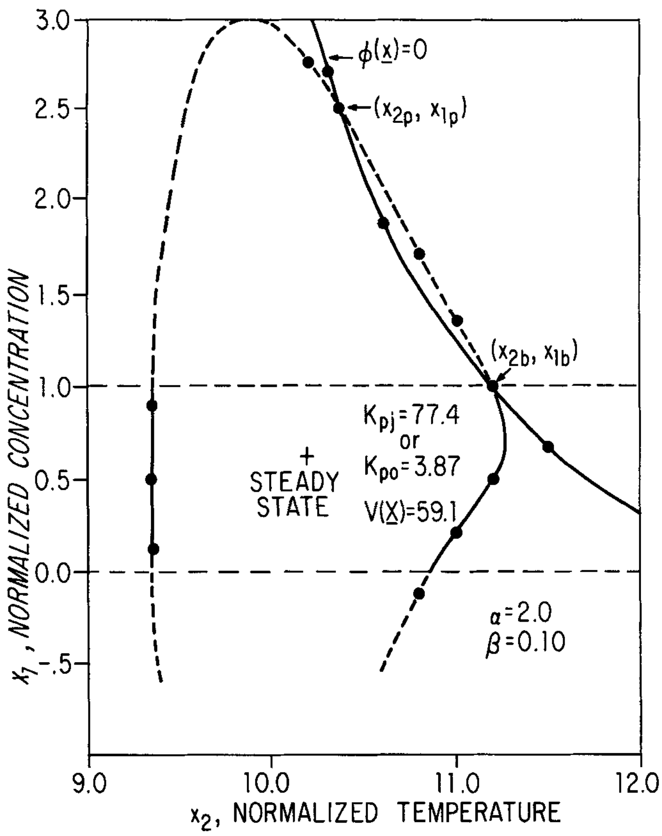

Fig. 6. Region of asymptotic stability for the example.

$$x_1 = \frac{-(2\alpha + k) \pm 2\sqrt{(\alpha + k)(2\alpha + \beta + \beta K_{pj})}}{\dfrac{Qk}{x_2^2}}$$

Choose the positive root to remain in the physical realizable part of the phase plane.

$$x_1 = \frac{[6.88\sqrt{(2 + k)} - (4 + k)]x_2^2}{185k} \qquad (20)$$

For the LF we have

$$[-x_1(2 + k) + 2.02]^2 + [kx_1 - 9.84x_2 + 99.5]^2 = 59.1 \qquad (21)$$

These two equations could be solved simultaneously by substituting the value of x_1 from Equation (20) into Equation (21). But, due to the exponential terms [contained in k from Equation (17)] that occur in this example, it is probably just as easy to plot a few points to determine the intersections. This procedure has the added advantage of showing the shapes of the curves. Information would also be needed for the stability range at other concentrations. This requires the solution of $V(\mathbf{X}) = 59.1$ at a few points for $0 < x_1 < 1.0$. The points and curves are given in Figure 6. Note that there are only two intersections of ϕ and V, and $x_{1b} < x_{1p}$. Also ϕ is monotonic and intersects $x_1 = 1.0$ at only one point. Thus the method is valid. Also observe that $K_{pj} = 77.4$ in this example gives a larger range of stable temperatures than any K_{pj} in Figure 1 where the natural boundary is not used. Figures 5 and 6 were obtained in the same manner as Figures 1 and 2.

Other types of control could be used, for example, proportional plus derivative negative feedback manipulating the feed temperature. For this control mode

$$X_o = -K_{po}(X_2 + K_{do}\dot{X}_2)$$

which gives the constraint equation

$$4\left(\alpha+\frac{\partial R}{C_o\partial X_1}\right)\left(\alpha+\beta-\frac{\partial R}{C_o\partial X_2}+\frac{\alpha K_{po}K_{do}}{1+\alpha K_{po}K_{do}}\left[\frac{\partial R}{C_o\partial X_2}-\alpha-\beta+\frac{1}{K_{do}}\right]\right)-\left(\frac{\partial R}{C_o\partial X_2}-\frac{\partial R}{C_o\partial X_1}\right)^2=0$$

and the LF

$$V(\mathbf{X})=\left(-\alpha X_1-\frac{R}{C_o}\right)^2+\left[\frac{R}{C_o}-X_2(\alpha+\beta)-\frac{\alpha K_{po}K_{do}}{1+\alpha K_{po}K_{do}}\left(X_2\left(\frac{1}{K_{do}}-\alpha-\beta\right)+\frac{R}{C_o}\right)\right]^2$$

The preceding procedure could be used if the appropriate criteria are fulfilled, but one of the control constants would have to be determined by some other information.

CONCLUSIONS

It has been shown that the region of asymptotic stability determined by a Liapunov function obtained by using Krasovskii's theorem can be extended if there exists a natural boundary of the system beyond which a trajectory cannot pass, and if this boundary lies inside the point of tangency between the Liapunov function and the constraint. Significant increases in the range of stable temperatures were obtained by considering the natural boundary in the case of the continuous stirred tank reactor. This method is not limited to Liapunov functions obtained by Krasovskii's theorem. Any method of obtaining these functions which gives a curve that bounds a region where $\dot{V}(\mathbf{X}) < 0$ and intersects a natural boundary of the system could be used.

Taking advantage of this point of intersection where the Liapunov function, the constraint, and the natural bound all have the same point in common, a design procedure was developed for a second-order nonlinear system and applied to the continuous flow, perfectly mixed, stirred-tank reactor. This information would be of benefit in the design of a controller for a nonlinear system by depicting the range of disturbances for which different controller gains would effect a return to the steady state. Two different approaches could be used. One could design by using linear approximations and then check to make sure the actual nonlinear system is stable for sufficiently large disturbances, or one could choose the controller gain directly by anticipating the largest disturbance.

It has been emphasized that in investigating the effects of altering controller gains, it is not sufficient to merely examine the change in the size of the $V(\mathbf{X}) < 0$ region. The change in the Liapunov function must also be considered.

A Liapunov function only furnishes sufficient conditions for asymptotic stability and thus the results and methods of this paper are conservative. The controllers have been assumed to be perfect; lags and deadtime have been omitted. However, the stability ranges obtained give a far clearer picture of the stability behavior of a system than can be achieved by a linear analysis which can only assure stability of a nonlinear system for sufficiently small disturbances, and can give no information about the region of asymptotic stability when the linearized system predicts continuous oscillation.

ACKNOWLEDGMENT

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NOTATION

A = frequency factor
a = heat transfer area
B = positive definite symmetric matrix
C = concentration
C_p = heat capacity
E = activation energy
$\mathbf{F}$ = Jacobian matrix
$\mathbf{f}(\mathbf{x})$ = arbitrary function (column vector)
$\mathbf{I}$ = unit matrix
K, L = constants
k = reaction rate temperature dependence
K_p = proportional control constant
K_d = derivative control constant
n = reaction order
Q = $E\rho C_p/R\Delta HC_o$
q = flow rate
r = reaction rate
R = reaction rate disturbance
R_g = gas constant
T = temperature
t = time
U = heat transfer coefficient
$V(\mathbf{X})$ = Liapunov function
v = reactor volume
$\mathbf{X}$ = disturbance column vector
$\mathbf{x}$ = generalized state column vector

Greek Letters

α = q/v
β = $Ua/\rho C_p v$
ΔH = exothermic heat of reaction
ρ = density
$\phi(\mathbf{x})$ = constraint that defines $\dot{V}(\mathbf{X}) < 0$ region

Superscripts and Subscripts

. = time derivative
– = vector or matrix
b = natural boundary
o = feed stream
j = reactor coolant
p = point of intersection other than at b
s = steady state
T = transpose of a matrix
1, 2 = components of vector

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